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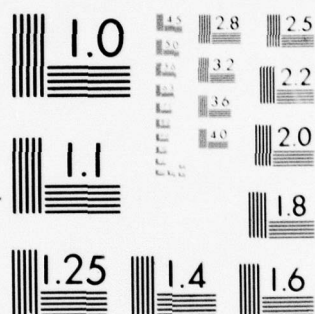
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Remarks on the Asymptotic Behavior of Solutions
to Damped Wave Equations in Hilbert Space.

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Abstract

Lower bounds are derived for the norms of solutions to a class of initial-value problems associated with the damped wave equation $u_{tt} + Au_t + Bu = 0$ in Hilbert space. Under appropriate assumptions on the linear operator B it is shown that even in the special strongly damped case where $A = \Gamma I$, $\Gamma > 0$, solutions are bounded away from zero as $t \rightarrow +\infty$, even when $\Gamma \rightarrow +\infty$.

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20. Abstract

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Abstract

Lower bounds are derived for the norms of solutions to a class of initial-value problems associated with the damped wave equation $u_{tt} + Au_t + Bu = 0$ in Hilbert space. Under appropriate assumptions on the linear operator B it is shown that even in the special strongly damped case where $A = \Gamma I$, $\Gamma > 0$, solutions are bounded away from zero as $t \rightarrow +\infty$, even when $\Gamma \rightarrow +\infty$.

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Various authors (and most notably Dafermos [1] and Russell [2]) have recently studied the asymptotic behavior of solutions to initial-value problems associated with damped linear wave equations of the form

$$(1) \quad u_{tt} + Au_t + Bu = 0 \quad \left(()_t = \frac{d}{dt} () \right)$$

where $u: [0, \infty) \rightarrow H$, a real Hilbert space with inner-product \langle, \rangle_H and natural norm $\|(\cdot)\|_H$; the usual assumptions which are made relative to (1) are that [2] A is a bounded linear operator on H which satisfies a coerciveness condition of the form

$$(2) \quad \langle v, Bv \rangle_H \geq \lambda \|v\|_H^2; \quad \lambda > 0, \quad v \in \mathcal{D}(B),$$

where $\mathcal{D}(B)$ denotes the domain of B and is such that $\overline{\mathcal{D}(B)} = H$. If $\langle Av, v \rangle_H \geq 0$ and A^{-1} exists (the strongly damped case) then it is well known that the energy

$$(3) \quad E(t) = \frac{1}{2} (\|u_t\|_H^2 + \langle u(t), Au(t) \rangle)$$

associated with (1) decays at a uniform exponential rate and the asymptotic stability of solutions to (1) follows immediately. Even if A^{-1} does not exist (the weakly damped case) it may still be shown that under various circumstances $\lim_{t \rightarrow \infty} E(t) = 0$. Our concern in this note will be with a special case of the strongly damped situation where $A = \Gamma I$, $\Gamma > 0$ a real number, but where the coerciveness condition (2) is not satisfied. We consider, in fact, the system

$$(4) \quad u_{tt}^{\alpha} + \Gamma u_t^{\alpha} - Nu^{\alpha} = 0$$

$$(5) \quad u^{\alpha}(0) = \alpha u_0, \quad u_t^{\alpha}(0) = v_0 \quad (u_0, v_0 \in \mathcal{D}(N))$$

where $\alpha > 0$ is a real number and $u^{\alpha} \in C^2([0, \infty); H)$ such that $u^{\alpha}(t) \in \mathcal{D}(N)$ for each $\alpha > 0, t \in [0, \infty)$. We have replaced the operator B in (1) by $-N$ in (4) so as to afford easy comparison with recent related work on growth estimates for solutions of undamped ($\Gamma = 0$) linear differential equations in Hilbert space [3], [4] which employ differential inequality arguments of the type we will use in this paper.

In the present situation the positivity assumption (2) would assume the form

$$(6) \quad \langle v, Nv \rangle_H \leq -\lambda \|v\|_H^2, \quad \lambda > 0, \quad v \in \mathcal{D}(N)$$

However, the assumptions that we will make here are that N is symmetric and that

$$(7a) \quad \langle v, Nv \rangle_H \geq 0, \quad v \in H$$

with there being (at least) one element $\bar{u}_0 \in \mathcal{D}(N)$ such that

$$(7b) \quad \langle \bar{u}_0, N\bar{u}_0 \rangle_H > 0$$

under the assumptions (7a) and (7b) it will be shown that solutions u^{α} of (4), (5) with $u_0 = \bar{u}_0$ and α sufficiently large satisfy

$$(8) \quad \lim_{t \rightarrow \infty} \|u^{\alpha}(t)\|_H^2 \geq \alpha^2 \|\bar{u}_0\|_H^2 e^{-\Sigma_0(\alpha, \Gamma)}$$

where $\Sigma_0(\alpha, \Gamma)$ depends on \bar{u}_0, v_0 and satisfies $\lim_{\Gamma \rightarrow \infty} \Sigma_0(\alpha, \Gamma) = 0$, for all $\alpha > 0$, so that

$$(9) \quad \lim_{\Gamma \rightarrow \infty} \lim_{t \rightarrow \infty} \|u^\alpha(t)\|_H^2 \geq \alpha^2 \|\bar{u}_0\|_H^2$$

for all α sufficiently large.

Remark In [1] a second real Hilbert space V is introduced with $\bar{V} = H, V \subset H$ algebraically and topologically. Letting V' denote the dual of V via \langle, \rangle_H , i.e.

$$(10) \quad \|v\|_{V'} = \sup_{w \in V} \frac{|\langle v, w \rangle_H|}{\|w\|_V}$$

the assumption in [1] which corresponds to (2) is that

$$(11) \quad \langle v, Bv \rangle_H \geq \lambda \|v\|_V^2, \quad \lambda > 0, v \in V$$

where $B \in L(V, V')$; such a setting is particularly appropriate for dealing with problems in linear partial differential equations and it will be obvious that our results carry over immediately in this framework also.

We now state and prove the basic growth estimate for the system (4), (5) from which the claimed asymptotic results (8) and (9) follow almost immediately; the argument presented below is a hybrid mixture of logarithmic concavity and logarithmic convexity arguments of the kind which have been used so successfully in recent years to establish results on uniqueness, stability, continuous dependence, and instability of solutions to initial-

value problems for abstract linear and nonlinear equations in Hilbert space [4], [5], [6], as well as various initial-boundary value problems for nonlinear partial differential equations [7], [8], [9], [10].

Theorem If $u^\alpha \in C^2([0, \infty); \mathcal{D}(N))$ is a solution of (4), (5), with $u_0 = \bar{u}_0$, and (7a), (7b) obtain, then

$$(12) \quad ||u^\alpha(t)||_H^2 \geq \alpha^2 ||\bar{u}_0||_H^2 \exp \left[\frac{\langle \bar{u}_0, v_0 \rangle_H}{\alpha ||\bar{u}_0||_H^2} (1 - e^{-\Gamma t}) \right]$$

for all $\alpha \geq ||v_0||_H / \sqrt{\langle \bar{u}_0, N\bar{u}_0 \rangle_H}$ and $t \geq 0$.

Proof Let $F_\alpha(t) = ||u^\alpha(t)||_H^2$, $0 \leq t < \infty$, where $u^\alpha \in C^2([0, \infty); H)$ is any solution of (4) (5) with $u_0 = \bar{u}_0$ and let $\beta > 0$ arbitrary. A direct computation, analogous to Levine([1], II, Thm I.), yields $(F'_\alpha \equiv \frac{d}{dt} F_\alpha)$

$$(13) \quad F_\alpha F_\alpha'' - (\beta+1)F_\alpha'^2 = 4(\beta+1)S_\alpha^2 + 2F_\alpha \{ \langle u^\alpha, u_{tt}^\alpha \rangle_H - (2\beta+1) \langle u_t^\alpha, u_t^\alpha \rangle_H \}$$

where

$$(14) \quad S_\alpha^2 = (\langle u^\alpha, u^\alpha \rangle_H) (\langle u_t^\alpha, u_t^\alpha \rangle_H) - \langle u^\alpha, u_t^\alpha \rangle_H^2 \geq 0,$$

by the Schwartz inequality, for all $\alpha > 0$; the result in (13) depends only on the form of $F_\alpha(t)$ and is independent of the particular equation satisfied by u_α . In view of (4) and (14) we may deduce from (13) the differential inequality

$$(15) \quad F_{\alpha} F_{\alpha}'' - (\beta+1) F_{\alpha}'^2 \geq 2 F_{\alpha} G_{\alpha, \beta}, \quad 0 \leq t < \infty$$

where for arbitrary $\beta > 0$

$$(16) \quad G_{\alpha, \beta} \equiv \langle u^{\alpha}, Nu^{\alpha} \rangle_H - \Gamma \langle u_t^{\alpha}, u_t^{\alpha} \rangle_H \\ - (2\beta+1) \langle u_t^{\alpha}, u_t^{\alpha} \rangle_H$$

Therefore,

$$(17) \quad G'_{\alpha, \beta} = - 4\beta \langle u_t^{\alpha}, Nu^{\alpha} \rangle_H - \Gamma \langle u_t^{\alpha}, u_t^{\alpha} \rangle_H \\ - \Gamma \langle u^{\alpha}, u_{tt}^{\alpha} \rangle_H + 2\Gamma(2\beta+1) ||u_t^{\alpha}||_H^2$$

but

$$(18) \quad G_{\alpha, \beta}(0) = \alpha^2 \langle \bar{u}_0, N\bar{u}_0 \rangle_H - \alpha \Gamma \langle \bar{u}_0, v_0 \rangle_H \\ - (2\beta+1) ||v_0||_H^2$$

so integration of (17) and substitution from (18) yields

$$(19) \quad G_{\alpha, \beta}(t) = G_{\alpha, \beta}(0) - 2\beta [\langle u^{\alpha}, Nu^{\alpha} \rangle_H - \alpha^2 \langle \bar{u}_0, N\bar{u}_0 \rangle_H] \\ - \Gamma [\langle u^{\alpha}, u_t^{\alpha} \rangle_H - \alpha \langle \bar{u}_0, v_0 \rangle_H] \\ + 2\Gamma(2\beta+1) \int_0^t ||u_{\tau}^{\alpha}||_H^2 d\tau \\ \geq (2\beta+1) [\alpha^2 \langle \bar{u}_0, N\bar{u}_0 \rangle_H - ||v_0||_H^2] \\ - 2\beta \langle u^{\alpha}, Nu^{\alpha} \rangle_H - \Gamma \langle u^{\alpha}, u_t^{\alpha} \rangle_H$$

where we have dropped the term proportional to $\int_0^t ||u_t^\alpha||_H^2 d\tau$.

However, in view (7) and the definition of F_α , if we now choose

$\alpha \geq ||v_0||_H / \sqrt{\langle \bar{u}_0, N\bar{u}_0 \rangle_H}$ then (19₂) may be replaced by

$$(20) \quad G_{\alpha,\beta}(t) \geq -2\beta \langle u^\alpha, Nu^\alpha \rangle_H - \frac{\Gamma}{2} F_\alpha'(t)$$

But, by (16)

$$(21) \quad \langle u^\alpha, Nu^\alpha \rangle_H = G_{\alpha,\beta} + \Gamma \langle u^\alpha, u_t^\alpha \rangle_H + (2\beta+1) ||u_t^\alpha||_H^2$$

and so substituting in (20) from (21) and solving for $G_{\alpha,\beta}$ we find that

$$(22) \quad G_{\alpha,\beta} \geq -\Gamma\mu(\beta)F_\alpha' - 2\beta ||u_t^\alpha||_H^2$$

where

$$(23) \quad \mu(\beta) \equiv (\beta + \frac{1}{2}) / (2\beta+1)$$

Now, suppose we take the inner-product of (4) with u^α , i.e.,

$$\langle u^\alpha, u_{tt}^\alpha \rangle_H + \Gamma \langle u^\alpha, u_t^\alpha \rangle_H - \langle u^\alpha, Nu^\alpha \rangle_H = 0$$

Then it is easily seen that this equation may be rewritten in the form

$$(24) \quad \frac{1}{2} \frac{d^2}{dt^2} ||u^\alpha||_H^2 + \frac{\Gamma}{2} \frac{d}{dt} ||u^\alpha||_H^2 = ||u_t^\alpha||_H^2 + \langle u^\alpha, Nu^\alpha \rangle_H$$

which yields, in view of (7a) and the definition of F , the inequality

$$(25) \quad ||u_t^\alpha||_H^2 \leq \frac{1}{2} F_\alpha'' + \frac{\Gamma}{2} F_\alpha'$$

Combining (22) and (25) we have

$$(26) \quad G_{\alpha, \beta} \geq -\Gamma(\beta + \mu(\beta))F_\alpha' - \beta F_\alpha''$$

and, therefore, in view of (15),

$$(27) \quad F_\alpha F_\alpha'' - (\beta + 1)F_\alpha'^2 \geq -2\Gamma(\beta + \mu(\beta))F_\alpha' F_\alpha - 2\beta F_\alpha F_\alpha''$$

or

$$(28) \quad F_\alpha F_\alpha'' - \left[\frac{\beta+1}{2\beta+1}\right] F_\alpha'^2 \geq -2\Gamma\gamma(\beta)F_\alpha' F_\alpha$$

where

$$(29) \quad \gamma(\beta) \equiv \frac{\beta + \mu(\beta)}{2\beta+1} = \frac{\beta + \left[\frac{\beta+\frac{1}{2}}{2\beta+1}\right]}{2\beta+1} = \frac{1}{2}$$

for all $\beta > 0$; therefore, (28) is identical with

$$(30) \quad F_\alpha F_\alpha'' - \left[\frac{\beta+1}{2\beta+1}\right] F_\alpha'^2 \geq -\Gamma F_\alpha' F_\alpha$$

for all $\beta > 0$. Taking the limit in (30) as $\beta \rightarrow 0$ we obtain the desired differential inequality, i.e.,

$$(31) \quad F_\alpha F_\alpha'' - F_\alpha'^2 \geq -\Gamma F_\alpha' F_\alpha$$

Direct integration of (30) yields

$$(32) \quad F_\alpha(t) \geq F_\alpha(0) \exp \left[\left\{ \frac{F_\alpha'(0)}{\Gamma F_\alpha(0)} \right\} (1 - e^{-\Gamma t}) \right]$$

and the desired estimate, i.e., (12) now follows directly from (5), with $u_0 = \bar{u}_0$, and the definition of F_α .

If we now set

$$(33) \quad \Sigma_0(\alpha, \Gamma) = \langle \bar{u}_0, v_0 \rangle_H / \alpha \Gamma ||\bar{u}_0||_H^2$$

then it is immediate from (12) that

$$(34) \quad ||u^\alpha(t)||_H^2 \geq \alpha^2 ||\bar{u}_0||_H^2 e^{-f(t)},$$

if $\alpha \geq ||v_0||_H / \sqrt{\langle \bar{u}_0, N\bar{u}_0 \rangle_H}$, where

$$(35) \quad f(t) \equiv \Sigma_0(\alpha, \Gamma)(e^{-\Gamma t} - 1)$$

satisfies $f(0) = 0$ and

$$(36) \quad f'(t) = -\Gamma \Sigma_0(\alpha, \Gamma) e^{-\Gamma t} \begin{cases} < 0, & \langle \bar{u}_0, v_0 \rangle_H > 0 \\ > 0, & \langle \bar{u}_0, v_0 \rangle_H < 0 \end{cases}$$

The asymptotic estimate claimed in (8) now follows directly from (33) - (35) if we let $t \rightarrow +\infty$ in (34) while the estimate delineated in (9) follows directly from (8) by taking the limit as $\Gamma \rightarrow \infty$ and using the definition (33) of $\Sigma_0(\alpha, \Gamma)$. Two other results are worth noting in passing, namely,

$$(37) \quad \lim_{\alpha \rightarrow +\infty} \left(\frac{||u^\alpha(t)||_H}{\alpha} \right)^2 \geq ||\bar{u}_0||_H^2, \text{ for all } t > 0$$

and, for $\alpha \geq ||v_0||_H / \sqrt{\langle \bar{u}_0, N\bar{u}_0 \rangle_H}$,

$$(38) \quad \lim_{t \rightarrow +\infty} ||u^\alpha(t)||_H^2 \geq \alpha^2 ||\bar{u}_0||_H^2 \exp\{-||v_0||_H / \alpha \Gamma ||\bar{u}_0||_H\}$$

which follows directly from (8), (33), and the Schwartz inequality.

As an application of the above results we may consider the equations of linear elasticity with viscous clamping, i.e.,

$$(39) \quad \ddot{u}_i^\alpha(x,t) + \Gamma \dot{u}_i^\alpha(x,t) - \frac{\partial}{\partial x_j} (c_{ijkl}(x) \frac{\partial u_k^\alpha(x,t)}{\partial x_\ell}) = 0$$

for $i = 1, 2, 3$, $x \in \Omega \subseteq \mathbb{R}^3$ (a bounded region with smooth boundary $\partial\Omega$) and $t \geq 0$; u is the elastic displacement vector while the $c_{ijkl}(x)$ are the components of the elasticities, $i, j, h, \ell = 1, 2, 3$, which are assumed to satisfy the usual symmetries, i.e., $c_{ijkl} = c_{jikl} = c_{klij}$. We take H to be the completion of $C_0^\infty(\Omega)$ with respect to

$$(40) \quad \langle \underline{u}, \underline{v} \rangle = \int_\Omega u_i v_i dx$$

i.e., $H = L_2(\Omega)$ and define

$$(41) \quad (Nu)_i = \frac{\partial}{\partial x_j} (c_{ijkl}(x) \frac{\partial u_k}{\partial x_\ell}), \quad u \in H_0^1(\Omega)$$

where $H_0^1(\Omega)$, the completion of $C_0^\infty(\Omega)$ with respect to

$$(42) \quad \langle u, v \rangle_+ = \int_\Omega \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} dx,$$

satisfies $H_0^1(\Omega) \subseteq L_2(\Omega)$ algebraically and topologically with $H_0^1(\Omega)$ dense in $L_2(\Omega)$. We note that N , as defined by (41) is bounded as a linear operator from $H_0^1(\Omega)$ into $H^{-1}(\Omega)$ where $H^{-1}(\Omega)$

is the completion of $C_0^\infty(\Omega)$ with respect to

$$(43) \quad ||w||_- = \sup_{v \in H_0^1(\Omega)} [|\int_{\Omega} w_i v_i dx| / (\int_{\Omega} \frac{\partial v_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} dx)^{1/2}]$$

We append to (39) initial and boundary datum of the form

$$(44a) \quad u_i^\alpha(x, t) = 0, \quad x \in \partial\Omega, \quad t \geq 0$$

$$(44b) \quad u_i^\alpha(x, 0) = \alpha f_i(x), \quad \dot{u}_i^\alpha(x, 0) = g_i(x),$$

where $f, g \in H_0^1(\Omega)$, and note that

$$(45) \quad \begin{aligned} \langle f, Nf \rangle &= \int_{\Omega} f_i \frac{\partial}{\partial x_j} (c_{ijkl} \frac{\partial f_k}{\partial x_l}) dx \\ &= - \int_{\Omega} c_{ijkl} \frac{\partial f_i}{\partial x_j} \frac{\partial f_k}{\partial x_l} dx \geq 0 \end{aligned}$$

provided the elasticities $c_{ijkl}(x)$ satisfy

$$(46) \quad c_{ijkl}(x) \xi_{ij} \xi_{kl} \leq 0 \quad (\text{all } \xi \in L(R^3; R^3) \text{ and } x \in \Omega)$$

If, in addition, there exists at least one element $\bar{f} \in H_0^1(\Omega)$ such that

$$(47) \quad \int_{\Omega} c_{ijkl}(x) \frac{\partial \bar{f}_i}{\partial x_j} \frac{\partial \bar{f}_k}{\partial x_l} dx < 0$$

then the results which follow from the theorem above may be applied to the system consisting of (39), (44a) and (44b), with $f = \bar{f}$, provided we choose

$$(48) \quad \alpha \geq (\int_{\Omega} g_i g_i dx)^{1/2} / (-\int_{\Omega} c_{ijkl} \frac{\partial \bar{f}_i}{\partial x_j} \frac{\partial \bar{f}_k}{\partial x_l} dx)^{1/2}.$$

References

1. Dafermos, C. M., "Wave Equations with Weak Damping", SIAM. J. Appl. Math. Vol. 18 (1970), 759-767.
2. Russell, D. L., "Decay Rates for Weakly Damped Systems in Hilbert Space obtained with Control-Theoretic Methods", J. Diff. Eqs., Vol. 19 (1975), 344-370.
3. Levine, H. A., "Uniqueness and Growth of Weak Solutions to Certain Linear Differential Equations in Hilbert Space", J. Diff. Eqs., Vol. 17 (1975), 73-81.
4. Knops, R. J. & L. R. Payne, "Growth Estimates for Solutions of Evolutionary Equations in Hilbert Space with Applications in Elastodynamics", Arch. Rat. Mech. Anal., Vol. 41 (1971) 363-398.
5. Levine, H. A., "Logarithmic Convexity, First Order Differential Inequalities and Some Applications", Trans. A.M.S., Vol. 152 (1970).
6. Levine, H. A., "Logarithmic Convexity and the Cauchy Problem for Some Abstract Second Order Differential Inequalities", J. Diff. Eqs., Vol. 8, (1970), 34-55.
7. Levine, H. A. & L. E. Payne, "Nonexistence Theorems for the Heat Equation with Nonlinear Boundary Conditions and for the Porous Medium Equation Backward in Time", J. Diff. Eqs., Vol. 16, (1974), 319-334.
8. Levine, H. A. & L. E. Payne, "Some Nonexistence Theorems for Initial-Boundary Value Problems with Nonlinear Boundary Constraints", Proc. A.M.S., Vol. 46, (1974), 277-284.
9. Levine, H. A., "Instability and Nonexistence of Global Solutions to Nonlinear Wave Equations of the form $\rho u_{tt} = -Au + F(u)$ ", Trans. A.M.S., Vol. 192, (1974), 1-21.
10. Levine, H. A., "Some Nonexistence and Instability Theorems for Formally Parabolic Equations of the Form $\rho u_t = Au + F(u)$ ", Arch. Rat. Mech. Anal., Vol. 51, (1973), 371-386.